# THE TROPICAL RANK OF A TROPICAL MATRIX

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ABSTRACT. In this paper we further develop the theory of matrices over the extended tropical semiring. Introducing a notion of tropical linear dependence allows for a natural definition of matrix rank in a sense that coincides with the notions of tropical regularity and invertibility.

## INTRODUCTION

One of the most important notions in linear algebra is the notion of rank, especially with a suitable relation to linear dependence. In the familiar tropical linear algebra the notion of dependence is absent, mostly since the ground max-plus semiring is idempotent. The special structure of the ***extended tropical semiring***, as introduced in [5], allows a natural definition for this absent notion, providing the tropical analogous to rank of matrices as in the classical theory, i.e. the maximal number of independent rows, and leading to the two important results:

- An $n \times n$ matrix $A$ has rank $n$ iff $A$ is tropically nonsingular iff $A$ is pseudo invertible,
- An $m \times n$ matrix $A$ has rank $k$ iff its maximal nonsingular minor is of size $k \times k$.

Although our framework is typically combinatorial, as these results show, the tropical analogous to classical results are carried naturally over the extended tropical semiring.

The main goal of this paper is a further development of the basics of tropical matrix algebra over the extended tropical semiring, $(\mathbb{T}, \oplus, \odot)$, as has been presented in [5]; we also use some of the terminology used in [6]. This extension is obtained by taking two copies of the reals,

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad \bar{\mathbb{U}} = \mathbb{R}^{\nu} \cup \{-\infty\},$$

each is enlarged by $\{-\infty\}$, and gluing them along $-\infty$ to define the set $\mathbb{T} = \bar{\mathbb{R}} \cup \bar{\mathbb{U}}$. We define the correspondence $\nu : \mathbb{R} \to \mathbb{U}$ to be the identity map, and denote the image of $a \in \mathbb{R}$ by $a^{\nu}$. Accordingly, elements of $\mathbb{U}$, which is called the ***ghost*** part of $\mathbb{T}$, are denoted as $a^{\nu}$; $\mathbb{R}$ is called the ***real*** part of $\mathbb{T}$. The map $\nu$ is sometimes extended to whole $\mathbb{T}$,

$$\nu : \mathbb{T} \longrightarrow \bar{\mathbb{U}}, \tag{1}$$

by declaring $\nu : a^{\nu} \mapsto a^{\nu}$ and $\nu : -\infty \mapsto -\infty$. (We use the generic notation $a, b \in \mathbb{R}$ and $x, y \in \mathbb{T}$.)

The set $\mathbb{T}$ is then provided with the following total order extending the usual order on $\mathbb{R}$:

(i) $-\infty \prec x$, $\forall x \in \mathbb{T}$;

(ii) for any real numbers $a < b$, we have $a \prec b$, $a \prec b^{\nu}$, $a^{\nu} \prec b$, and $a^{\nu} \prec b^{\nu}$;

(iii) $a \prec a^{\nu}$ for all $a \in \mathbb{R}$.

Then, $\mathbb{T}$ is endowed with the two operations $\oplus$ and $\odot$ , defined as follows:

$$\begin{aligned} x \oplus y &= \begin{cases} max_{(\prec)}\{x, y\}, & x \neq y, \\ x^{\nu}, & x = y \neq -\infty, \end{cases} \\ -\infty \oplus -\infty &= -\infty, \end{aligned} \qquad \begin{aligned} a \odot b &= a + b, \\ a^{\nu} \odot b &= a \odot b^{\nu} = a^{\nu} \odot b^{\nu} = (a+b)^{\nu}, \\ (-\infty) \odot x &= x \odot (-\infty) = -\infty. \end{aligned}$$

We usually write $xy$ for the product $x \odot y$, for short. Similarly, the division is written $\frac{x}{y}$, for $y \neq -\infty$.

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Triple $(\mathbb{T}, \oplus, \odot)$ is called the extended tropical semiring; this semiring is nonidempotent commutative semiring, since $a \oplus a = a^\nu$, with the unit element $\mathbb{1}_\mathbb{T} := 0$ and the zero element $\mathbb{0}_\mathbb{T} := -\infty$. This, and the fact that $(\mathbb{R}, \odot)$ is a group and $(\bar{\mathbb{U}}, \oplus, \odot)$ is an ideal, provides $\mathbb{T}$ with a more richer structure to which much of the theory of commutative algebra can be transferred.

The connection with the standard tropical (max-plus) semiring is established by the natural semiring epimorphism,

$$\pi : (\mathbb{T}, \oplus, \odot) \longrightarrow (\mathbb{R} \cup \{-\infty\}, \max, + ), \tag{2}$$

where $\pi : a^\nu \mapsto a$, $\pi : a \mapsto a$ for all $a \in \mathbb{R}$, and $\pi : -\infty \mapsto -\infty$. (We write $\pi(x)$ for the image of $x \in \mathbb{T}$ in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$.) This epimorphism induces epimorphisms of polynomial semirings, Laurent polynomial semirings, and tropical matrices.

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## 1. Tropical vector spaces

As in the classical ring theory, the tropical space $\mathbb{T}^{(n)}$, consisting of all $n$-tuples $(x_1, \dots, x_n)$ with entries $x_i \in \mathbb{T}$, is treated as a semiring module with addition, and multiplication by $\alpha \in \mathbb{T}$, defined with respect to $(\mathbb{T}, \oplus . \odot)$. An $n$-tuple $(x_1, \dots, x_n) \in \mathbb{T}^{(n)}$ is called vector, and a vector having only ghost, or $-\infty$, entries (i.e. $(x_1, \dots, x_n) \in \bar{\mathbb{U}}^{(n)}$) is termed ***ghost vector***.

**Definition 1.1.** *A set $W = \{e_1, \dots, e_n\} \subset \mathbb{T}^{(n)}$ is a **classical base** of $\mathbb{T}^{(n)}$, if every element of $\mathbb{T}^{(n)}$ can be written uniquely in the form $\bigoplus_{i=1}^n \alpha_i e_i$, where $\alpha_i \in \mathbb{T}$.*

The ***standard base*** of $\mathbb{T}^{(n)}$ is defined as

$$e_1 = (0, -\infty, \dots, -\infty), \quad e_2 = (-\infty, 0, -\infty, \dots, -\infty), \quad \dots, \quad e_n = (-\infty, -\infty, \dots, 0).$$

**Definition 1.2.** *A collection of vectors $v_1, \dots, v_m$ is said to be **tropically dependent** if there exist $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{R}}$, but not all of them $-\infty$, for which*

$$\alpha_1 v_1 \oplus \dots \oplus \alpha_m v_m \in \bar{\mathbb{U}}^{(n)},$$

*otherwise the vectors are said to be **tropically independent**. We call these $\alpha_i$'s the **dependence coefficients** of $v_1, \dots, v_m$.*

Any set of vectors containing a ghost vector is tropically dependent; in particular, a singleton consisting of a ghost vector is tropically dependent.

**Example 1.3.** *Let $v_1, v_2, v_3$, and $v_4$ be the following tropical vectors:*

$$v_1 = (0, 1), \qquad v_2 = (1, 2), \qquad v_3 = (2, 0), \qquad v_4 = (2^\nu, 0).$$

*Then, $v_1$ and $v_2$ are tropically dependent, since $1v_1 \oplus v_2 \in \bar{\mathbb{U}}^{(2)}$. The vectors $v_1$ and $v_3$ are tropically independent, but $v_1$ and $v_4$ are tropically dependent, i.e. $1v_1 \oplus v_4 \in \bar{\mathbb{U}}^{(2)}$.*

Different from the classical theory, where the ground structure is a field, in which the notions of linear dependence and span coincide, these notions do not coincide in the tropical framework. Namely, even if a collection of vectors is linearly dependent it might happen that no one can be expressed in terms of other vectors; for example, take

$$v_1 = (1, 1, -\infty), \quad v_2 = (1, -\infty, 1), \quad \text{and} \quad v_3 = (-\infty, 1, 1),$$

these vectors are linearly dependent, i.e. $v_1 \oplus v_2 \oplus v_3 \in \bar{\mathbb{U}}^{(3)}$, but non of these vectors can be written in terms of the others.

## 2. Regularity of tropical matrices

2.1. **Tropical matrices.** It is standard that since $\mathbb{T}$ is a semiring then we have the semiring $M_{n\times n}(\mathbb{T})$ of $n \times n$ matrices with entries in $\mathbb{T}$, where addition and multiplication are induced from $\mathbb{T}$ as in the familiar matrix construction. The ***unit*** element $I$ of $M_{n\times n}(\mathbb{T})$, is the matrix with 0 on the main diagonal and whose off-diagonal entries are $-\infty$; the ***zero*** matrix is $Z = (-\infty)I$; therefore, $M_{n\times n}(\mathbb{T})$ is also a multiplicative monied.

We write $A = (a_{i,j})$ for a tropical matrix in $M_{n\times n}(\mathbb{T})$ and denote the entries of $A$ as $a_{i,j}$. We say that $A$ is **real matrix** if each $a_{i,j}$ is in $\bar{\mathbb{R}}$, $A$ is called **ghost matrix** when each $a_{i,j}$ belong ot $\bar{\mathbb{U}}$. Since $\mathbb{T}$ is a commutative semiring, $xA = Ax$ for any $x \in \mathbb{T}$. (We denote the set of $m \times n$ matrices by $M_{m\times n}(\mathbb{T})$.)

As in the familiar way, we define the ***transpose*** of $A = (a_{i,j})$ to be $A^{\mathrm{t}} = (a_{j,i})$. The ***minor*** $A_{i,j}$ is obtained by deleting the $i$ row and $j$ column of $A$. We define the ***tropical determinant*** to be

$$|A| = \bigoplus_{\sigma\in S_n} \left(a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right), \tag{3}$$

where $S_n$ is the set of all the permutations on $\{1, \dots, n\}$. Equivalently, the tropical determinant $|A|$ can be written also in terms of minors as

$$|A| = \bigoplus_{j} a_{i,j} |A_{i,j}|, \tag{4}$$

for some fixed index $i$. Indeed, in the classical terminology, since parity of indices' sums are not involved, the tropical determinant is a permanent, what makes the tropical determinant a pure combinatorial function.

We use the notation $\hat{\sigma}$ for a permutation, not necessarily unique, whose $\nu$-evaluation in $A$ equals $|A|^{\nu}$, and write

$$\gamma = a_{1,\hat{\sigma}(1)} \cdots a_{n,\hat{\sigma}(n)};$$

therefore $\pi(\gamma) = \pi(|A|)$, or equivalently $\gamma^{\nu} = (|A|)^{\nu}$. (We use both forms for convenience.) We say that two permutations $\sigma_1$ and $\sigma_2$ in $S_n$ are **disjoint** if $\sigma_1(i) \neq \sigma_2(i)$ for each $i = 1, \dots, n$.

**Remark 2.1.** *The tropical determinant has the following properties:*

(1) *Transposition and reordering of rows or columns leave the determinant unchanged;*

(2) *The determinant is linear with respect to scalar multiplication of any given row or column by a real.*

The ***adjoint*** matrix $\mathrm{adj}(A)$ of a matrix $A = (a_{i,j})$ is defined as the matrix $(a'_{i,j})^{\mathrm{t}}$ where $a'_{i,j} = |A_{j,i}|$.

**Definition 2.2.** *A matrix $A \in M_{n\times n}(\mathbb{T})$ is said to be* ***tropically singular****, or singular, for short, whenever $|A| \in \bar{\mathbb{U}}$, otherwise $A$ is called* ***tropically nonsingular****, or nonsingular, for short.*

In particular, when two or more different permutations, $\hat{\sigma}_1, \hat{\sigma}_2, \dots \in S_n$, achieve the $\nu$-value of $|A|$ simultaneously, or the permutation $\hat{\sigma}$ that reaches the $\nu$-value of $|A|$ involves an entry in $\bar{\mathbb{U}}$, then $A$ is singular.

**Remark 2.3.** *In this combinatorial view, for real matrices, our definition of singularity coincides with the known definition for matrices over $(\bar{\mathbb{R}}, \max, +)$, cf.* [2].

To establish a notion of pseudo invariability for $M_{n\times n}(\mathbb{T})$, viewed as monoid, we define a ***pseudo unit matrix*** to be a **regular** matrix with $0$ on the main diagonal and whose off-diagonal entries are in $\bar{\mathbb{U}}$; in particular, the unit matrix is also a pseudo unit. We use these pseudo unit matrices to define the distinguished subset $U_{n\times n}(\mathbb{T}) \subset M_{n\times n}(\mathbb{T})$ as

$$U_{n\times n}(\mathbb{T}) = \left\{ \widetilde{I} \;:\; \widetilde{I} \;\text{ is a pseudo unit matrix} \right\}. \tag{5}$$

**Definition 2.4.** *A matrix $A \in M_{n\times n}(\mathbb{T})$ is said to be* ***pseudo invertible*** *if there exits a matrix $B \in M_{n\times n}(\mathbb{T})$ such that $AB \in U_{n\times n}(\mathbb{T})$ and $BA \in U_{n\times n}(\mathbb{T})$. If $A$ is pseudo invertible, then we call $B$ a* ***pseudo inverse matrix*** *of $A$ and denote it as $A^{\triangledown}$.*

Having this setting, we state one of our main theorems analogues to the classical relation, [5, Theorem 3.3]:

**Theorem 2.5.** *A matrix $A \in M_{n\times n}(\mathbb{T})$ is pseudo invertible iff is tropically regular. In case $A$ is regular, $A^{\triangledown}$ can be defined as $A^{\triangledown} = \frac{\mathrm{adj}(A)}{|A|}$, and is called the* ***canonical pseudo inverse*** *of $A$.*

2.2. **Lemmas on tropical regularity.** Our main computational tool in tropical matrix theory is the **weighted digraph** $G = (V, E)$ of an $n \times n$ matrix $A = (a_{i,j})$, which is defined to have vertex set $V = \{1, \dots, n\}$, and an edge $(i, j)$ from $i$ to $j$ (given **weight** $a_{i,j}$) whenever $a_{i,j} \neq -\infty$. We denote this graph by $G_A$. In this view, reordering of rows or columns of $A$ is equivalent to relabeling of vertices on $G_A$.

We use [3] as a general reference for graphs. We always assume that $V = \{v_1, \dots, v_n\}$, for convenience of notation. The **out-degree**, $d_{\text{out}}(v)$, of a vertex is the number of edges emanating from $v$, and the **in-degree**, $d_{\text{in}}(v)$, is the number edges terminating at $v$. A **sink** is a vertex with $d_{\text{out}}(v) = 0$, while a **source** is a vertex with $d_{\text{in}}(v) = 0$.

The **weight** $w(P)$ of a path $P$ is defined to be the sum of the weights of the edges comprising $P$, counting multiplicity. A **simple cycle** is a simple path for which $d_{\text{out}}(v) = d_{\text{in}}(v) = 1$ for every vertex $v$ of the path; thus, the initial and terminal vertices are the same. A simple cycle of length 1 is then a loop. We define a $k$-**multicycle** $C$ in a digraph to be the union of disjoint simple cycles, the sum of whose lengths is $k$.

Writing a permutation $\sigma$ as a product $\mu_1 \cdots \mu_t$ of disjoint cyclic permutations, we see that each permutation $\sigma$ corresponds to an $n$-multicycle in $G_A$, and their highest weight matches $|A|$. In particular, when $|A| \in \mathbb{R}$, there is a unique $n$-multicycle having highest weight. Conversely, any $n$-multicycle corresponds to a permutation on $A$.

**Remark 2.6.** *Given a digraph $G$ where $d_{\text{in}}(v) \geq 1$ (resp. $d_{\text{out}}(v) \geq 1$), for each $v \in V$, then $G$ contains a simple cycle. Indeed, otherwise $G$ must have a source (resp. sink), $v \in V$, in contradiction to $d_{\text{in}}(v) \geq 1$ (resp. $d_{\text{out}}(v) \geq 1$), respectively.*

In the following exposition we write $A \preceq 0^\nu$ for a matrix $A$ all of whose entries are $\preceq 0^\nu$ and assume $|A| \neq -\infty$.

**Lemma 2.7.** *Assume $A \preceq 0^\nu$ is an $n \times n$ matrix, each of whose columns (resp. rows) contains at least one 0-entry or $0^\nu$-entry, then, for some $i$, $a_{i,\hat{\sigma}(i)} \in \{0, 0^\nu\}$.*

*Proof.* We may assume $\hat{\sigma} \in S_n$ is the identity. Suppose $\hat{\sigma}$ does not involve any 0-entry or $0^\nu$-entry, and let $G'_A$ be the reduced graph of $A$ obtained by erasing all edges of $G_A$ having weights $\prec 0$. Thus, $G'_A$ has no self loops and $d_{\text{out}}(v_i) \geq 1$ for each $v_i \in V$. But, by Remark 2.6, $G'_A$ has a simple cyclic $C = (v_{i_1,i_2}, \dots, v_{i_{k-1},i_k}, v_{i_k,i_1})$ of wight 0 or $0^\nu$ which contradicts the maximality of $|A|$, since $w(v_{i_1,i_2}, \dots, v_{i_{k-1},i_k}, v_{i_k,i_1}) \succ w(v_{i_1,i_1}, \dots, v_{i_k,i_k})$. □

**Corollary 2.8.** *An $n \times n$ matrix $A \preceq 0^\nu$, each of whose columns (resp. rows) contains at least one $0^\nu$-entry, is singular.*

**Lemma 2.9.** *An $n \times n$ matrix $A \preceq 0^\nu$, each of whose columns (resp. rows) contains either, at least two 0-entries or a $0^\nu$-entry, is singular.*

*Proof.* We may assume $\hat{\sigma}$ is the identity. If $a_{i,i} = 0^\nu$, for some $i$, we are done. Otherwise, let $G'_A$ be the reduced graph of $A$ obtained by erasing all edges of $G_A$ having weights $\prec 0$, i.e. $G'_A$ has only edges weighted 0 or $0^\nu$. Considering an edge with wight $0^\nu$ as a duplicated edge, $d_{\text{in}}(v_i) \geq 2$, for each vertex of $G'_A$. Thus, by Lemma 2.7, $G'_A$ has a self-loop.

Erase all the self-loops of $G'_A$ (in particular, non of these self-loops is a duplicated edge), and denote the new graph by $G''_A$. Each vertex of $G''_A$ has $d_{\text{in}}(v_i) \geq 1$ and thus, by Remak 2.6, $G''_A$ has a simple cycle, say $C = (v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_{i_1})$. This means that $G'_A$ must have a self-loop for each $v_{i_u}$, $u = 1, \dots, k$, since otherwise we would get a contradiction to the maximality of $\gamma$, that is the $\nu$-evaluation of $\hat{\sigma}$ in $A$. Thus, the permutation obtained from $\hat{\sigma}$ by replacing these self-loops with the simple cycle $C$ has the same $\nu$-evaluation $\gamma$ as $\hat{\sigma}$ has. □

**Remark 2.10.** *Any $n \times n$ singular matrix $A$ has an $n \times (n-1)$ submatrix which can be replaced by $\pi(A_{i,j})$ without changing the singularity of $A$. Indeed, given a permutation $\hat{\sigma} \in S_n$, let $\gamma$ denote the product $a_{1,\hat{\sigma}(1)} \cdots a_{n,\hat{\sigma}(n)}$, then:*

1. *If $\gamma \in \bar{\mathbb{U}}$ then there is some $a_{i,\hat{\sigma}(i)} \in \bar{\mathbb{U}}$ and we can replace all the columns $j \neq \hat{\sigma}(i)$. (Note that if $\gamma = -\infty$ then $|A| = -\infty$.)*
2. *When $\gamma \in \mathbb{R}$, since $A$ is singular, there are at least two different permutations $\hat{\sigma}_1$ and $\hat{\sigma}_2$, so we can replace any possible $n \times (n-1)$ submatrix of $A$.*

## 3. The rank of tropical matrices

The notion of linear dependence, cf. Definition 1.2, provides a natural definition for the rank of a tropical matrix:

**Definition 3.1.** *The* ***tropical rank****, denoted* $\mathrm{rk}(A)$*, of a matrix* $A$ *is defined to be the maximal number of tropically independent rows in* $A$*.*

By this definition a nonzero matrix, i.e. $A \neq (-\infty)$, can have rank 0; for example any matrix all of whose entries are ghost has rank 0.

The following familiar properties of matrix rank are easily checked for our tropical rank:

(i) The rank of a submatrix can not exceed the rank of the whole matrix.

(ii) The rank is invariant under reordering of either rows or columns.

(iii) The rank is invariant under (tropical) multiplication of rows or columns by real constants, and under insertion of a row or a column obtained as a combination of others.

The last property is true, since otherwise if it would changed the rank, then one could a priori choose this combination to obtain a lower rank. On the other hand, just as in the classical theory, vectors which are tropically dependent in the initial collection are still tropically dependent in any extended collection.

Later, we prove that $\mathrm{rk}(A)$ is also equal to the maximal number of its independent columns, and therefore that the tropical rank of a matrix and its transpose are the same, cf. Corollary 3.13.

### 3.1. **Tropical regularity and tropical dependence.**

The theorems in this subsection provide the connection between tropical dependence and tropical regularity, we open with the special case of matrices having determinant equals $-\infty$.

**Definition 3.2.** *We say that a set* $r_1, \dots, r_m$ *of rows has* ***rank defect*** $k$ *if there are* $k$ *columns, which we denote as* $c_1, \dots, c_k$*, such that* $a_{i,j} = -\infty$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq k$*.*

For example, the rows $(2, -\infty, 2, -\infty), (-\infty, -\infty, -\infty, 2), (1, -\infty, -\infty, -\infty)$ have rank defect 1, since they are all $-\infty$ in the second column.

**Proposition 3.3.** *Given an* $n \times n$ *matrix* $A$*, then* $|A| = -\infty$*, iff for some* $1 \leq k \leq n$*,* $A$ *has* $k$ *rows having rank defect* $n + 1 - k$*.*

*Proof.* ($\Leftarrow$) If $k = n$ then this is obvious, since some column is entirely $-\infty$. If $n > k$, we take one of the columns $c_j$ other than $c_1, \dots, c_k$ of Definition 3.2. Then for each $i$, the $(i, j)$ minor $A_{i,j}$ has at least $k - 1$ rows with rank defect $(n - 1) + 1 - k$, so has determinant $-\infty$ by induction; hence $|A| = -\infty$, by Formula (4).

($\Rightarrow$) We are done if all entries of $A$ are $-\infty$, so assume for convenience that $a_{n,n} \neq -\infty$. Then $|A_{n,n}| = -\infty$, so, by induction, $A_{n,n}$ has $k \geq 1$ rows of rank defect $(n - 1) + 1 - k = n - k$. We may assume that $a_{i,j} = -\infty$ for $1 \leq i \leq k$ and $1 \leq j \leq n - k$. Thus, we can partition $A$ as the matrix

$$A = \begin{pmatrix} -\infty & B' \\ B'' & C \end{pmatrix},$$

where $-\infty$ denotes the $k \times n - k$ matrix all of whose entries $-\infty$, $B'$ is a $k \times k$ matrix, $B''$ is an $n - k \times n - k$ matrix, and $C$ is an $n - k \times k$ matrix. Accordingly, $|B'| = -\infty$ or $|B''| = -\infty$.

If $|B'| = -\infty$, then, by induction, $B'$ has $k'$ rows of rank defect $k + 1 - k'$, thus, the same $k'$ rows in $A$ have rank defect $(n - k) + k + 1 - k' = n + 1 - k'$, and we are done taking $k'$ instead of $k$. If $|B''| = -\infty$, then, by induction, $B''$ has $k''$ rows of rank defect $(n - k) + 1 - k''$, these $k + k''$ rows in $A$ have rank defect $n + 1 - (k + k'')$, and we are done, taking $k + k''$ instead of $k$. $\square$

**Theorem 3.4.** *An* $n \times n$ *tropical matrix of rank* $< n$ *is tropically singular.*

*Proof.* Let $r_i$ denotes the $i$'th row of $A$. Since $\mathrm{rk}(A) < n$, there are $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{R}}$, not all of them $-\infty$, such that $\alpha_1 r_1 \oplus \dots \oplus \alpha_n r_n \in \bar{\mathbb{U}}^{(n)}$.

If $\alpha_i = -\infty$ for some $i$, say $i = n$, then the first $(n - 1)$ rows are tropically dependent and each minor $A_{n,j}$ has rank $< (n - 1)$. But then, by induction, each $A_{n,j}$ is singular and, by Formula (4), $|A| = \bigoplus_j a_{n,j} \, |A_{n,j}| \in \bar{\mathbb{U}}$.

Assuming all $\alpha_i$'s are in $\mathbb{R}$, we replace each row $r_i$ of $A$ by $\alpha_i r_i$ and have $\bigoplus_i r_i \in \bar{\mathbb{U}}^{(n)}$. Let $\beta_i$ denotes the maximal value in each column $j$. If $\beta_i = b_i^\nu$ we take $b_i$ instead of $\beta_i$; when $\beta_j = -\infty$, for some $j$, we replace it by an arbitrary real. Let $A'$ be the matrix obtained by dividing each column $j$ of $A$ by $\beta_j$, accordingly $A'$ satisfies the conditions of Lemma 2.9 and is singular. Since regularity/singularity is preserved under the above operations, $A$ is singular. $\square$

**Example 3.5.** *Consider a $2 \times 2$ matrix $(a_{i,j})$ with rank $= 1$. Then, there are $\alpha_1, \alpha_2 \in \bar{\mathbb{R}}$ such that $\alpha_1(a_{1,1}, a_{1,2}) \oplus \alpha_2(a_{2,1}, a_{2,2}) \in \bar{\mathbb{U}}^{(2)}$. Note that $\alpha_i \neq -\infty$, since otherwise for $k \neq i$ $(a_{k,1}, a_{k,2}) \in \bar{\mathbb{U}}^{(2)}$ which would contradict the data that $\mathrm{rk}(A) = 1$. Replacing each row $r_i$ of $A$ by $\alpha_i r_i$ and expanding the determinant we get*

$$(\alpha_1 a_{1,1})(\alpha_2 a_{2,2}) = (\alpha_2 a_{1,2})(\alpha_2 a_{2,1}) \quad \Longrightarrow \quad \alpha_1\alpha_2|A| = \alpha_1\alpha_2(a_{1,1}a_{2,2} \oplus a_{1,2}a_{2,1}) \in \bar{\mathbb{U}},$$

*i.e. $A$ is tropically singular.*

**Theorem 3.6.** *An $n \times n$ matrix $A$ has rank $< n$ iff $A$ is tropically singular.*

*Proof.* $(\Rightarrow)$ By Theorem 3.4.

$(\Leftarrow)$ Assuming that $A$ is singular we need to prove that the rows of $A$ are tropically dependent. Since parts of the proof is by induction $n$, the size of $A$, we assume the theorem is true for $(n-1)$; the case of $n = 1$ obvious. (The case of $n = 2$ is provided in Example 3.8.)

Throughout this prove, we assume $\hat{\sigma}$ is the identity i.e.

$$(|A|)^\nu = \gamma^\nu = (a_{1,1} \cdots a_{n,n})^\nu,$$

this hypothesis is not affected by multiplying through any row or column by a given $\alpha \in \mathbb{R}$. We also remark that when determining the dependence coefficients $\alpha_i$'s, we may assume the relevant $a_{i,j}$ are in $\bar{\mathbb{R}}$, since otherwise for $a_{i,j} = b^\nu$ we take $b$ instead.

**Case I:** For notational convenience, if $A$ has an $m \times m$ singular submatrix $A'$ with $\pi(|A'|) = \pi(a_{i_1,i_1} \cdots a_{i_m,i_m})$, renumbering the indices, we assume that the singular submatrix $A'$ with the minimal $m$ is the upper left submatrix of $A$, in particular if $a_{i,i} \in \bar{\mathbb{U}}$, for some $i$, renumbering the indices we may assume $a_{1,1} \in \bar{\mathbb{U}}$.

Let

$$\alpha_i = \pi\left(|A_{i,1}|\right), \tag{6}$$

excluding the case when all $\alpha_i$'s are $-\infty$, see Case II, we claim that $\bigoplus_i \alpha_i r_i \in \bar{\mathbb{U}}^{(n)}$, i.e.

$$\bigoplus_i \alpha_i a_{i,j} \in \bar{\mathbb{U}}, \qquad \text{for each } j = 1, \dots, n. \tag{7}$$

Suppose $j = 1$, then $\bigoplus_i \alpha_i a_{i,1} \in \bar{\mathbb{U}}$, since this is just the expansion of $|A|$ along the first column of $A$, i.e. $\bigoplus_i \alpha_i a_{i,1} = |A|$. Indeed, if $m = 1$, i.e. $a_{1,1} \in \bar{\mathbb{U}}$, we are done. Otherwise, $(a_{1,1}|A'_{1,1}|)^\nu = (a_{1,i}|A'_{1,i}|)^\nu = |A'|$ for some $1 < i \leq m$. Thus, since $\gamma = a_{1,1}|A'_{1,1}|\beta = a_{1,i}|A'_{1,i}|\beta$, up to $\nu$, we have $\alpha_1 a_{1,1} = \alpha_i a_{1,i}$.

Assume $\alpha_\ell a_{\ell,j}$, with $j > 1$, is a component with maximum $\nu$-value in the sum $\alpha_1 a_{1,j} \oplus \cdots \oplus \alpha_n a_{n,j}$. If $\alpha_\ell a_{\ell,j} = -\infty$ we are done, otherwise $\alpha_\ell$ and $a_{\ell,j}$ are not $-\infty$, then

$$\alpha_\ell = \pi\left(|A_{\ell,1}|\right) = \pi\left(a_{1,\sigma(1)} \cdots a_{\ell-1,\sigma(\ell-1)} a_{\ell+1,\sigma(\ell+1)} \cdots a_{n,\sigma(n)}\right), \qquad \sigma(i) \neq 1,$$

for some $\sigma \in S_n$. Let $u$ be the index for which $\sigma(u) = j$, i.e. $u \neq \ell$, then

$$\frac{|A_{\ell,1}|}{a_{u,j}} = \frac{\left(a_{1,\sigma(1)} \cdots a_{u,j} \cdots a_{n,\sigma(n)}\right)}{a_{u,j}} = \left(a_{1,\sigma(1)} \cdots a_{u-1,\sigma(u-1)} a_{u+1,\sigma(u+1)} \cdots a_{n,\sigma(n)} a_{\ell,j}\right),$$

up to $\nu$, which must be equal to

$$\frac{|A_{u,1}|}{a_{\ell,j}} = \frac{\left(a_{1,\sigma'(1)} \cdots a_{u-1,\sigma'(u-1)} a_{u+1,\sigma'(u+1)} \cdots a_{\ell,\sigma'(\ell)} \cdots a_{n,\sigma'(n)} a_{\ell,j}\right)}{a_{\ell,j}},$$

since otherwise $a_{u,j}|A_{u,1}| \succ a_{\ell,j}|A_{\ell,1}|$, contrary to hypothesis. So, $a_{u,j}|A_{u,1}|$ and $a_{\ell,j}|A_{\ell,1}|$ are two different terms in Formula (7) having a same $\nu$-value.

**Case II:** When $|A| = -\infty$, with all $\alpha_i = |A_{i,1}|$ are $-\infty$, we take $m$ maximal such that $A$ has an $m \times m$ submatrix of determinant $\neq -\infty$, and let $\gamma$ denote the determinant of the $m \times m$ submatrix $A_m$ of $A$

of maximal $\nu$-value. By induction, we may assume that $m = n-1$. Furthermore, it is enough to find a dependence among the $k$ rows obtained in Proposition 3.3, so, again, by induction, we may assume that $k = n$, and the entries in the first column are all $-\infty$. Since $a_{1,1} = -\infty$ and $|A_m| \neq -\infty$, namely $A$ has an $(n-1)\times(n-1)$ minor whose determinant $\neq -\infty$, the proof is then completed by the same arguments of Case I. $\square$

**Corollary 3.7.** *A matrix $A \in M_{n\times n}(\mathbb{T})$ has rank $n$ iff $A$ is non-singular iff $A$ is pseudo inevitable.*

*Proof.* The proof is derived from Theorem 3.6 and Theorem 2.5. $\square$

This corollary provides the complete tropical analogues to the well known classical relations between regularity, invertibility, and rank of matrices.

**Example 3.8.** *Suppose $A = (a_{i,j})$ is a $2\times 2$ singular matrix, i.e. $|A| = a_{1,1}a_{2,2} \oplus a_{1,2}a_{2,1}$. If $|A| = -\infty$ then $A$ has a $-\infty$ row, say $r_1$, then set $\alpha_2 = -\infty$ and take arbitrary real $\alpha_1$. Otherwise, $\hat{\sigma}$ is the identity, so take*

$$\alpha_1 = \pi(|A_{1,1}|) = \pi(a_{2,2}), \qquad \textit{and} \qquad \alpha_2 = \pi(|A_{2,1}|) = \pi(a_{1,2}).$$

*Then,*

$$\alpha_1 r_1 \oplus \alpha_1 r_1 = \pi(a_{2,2})(a_{1,1}, a_{1,2}) \oplus \pi(a_{1,2})(a_{2,1}, a_{2,2}) \in \bar{\mathbb{U}}^{(2)}.$$

**Example 3.9.** *Consider the matrix*

$$A = \begin{pmatrix} 1 & 4 & -1 \\ 1 & 0 & 6 \\ -4 & 1 & 3 \end{pmatrix}$$

*whose determinant equal $8^{\nu}$, and thus is singular. The tropical dependence of the rows of $A$ is given by $\alpha_1 = |A_{1,1}| = 7$, $\alpha_2 = |A_{2,1}| = 7$, and $\alpha_3 = |A_{3,1}| = 10$.*

**Theorem 3.10.** *Any $k > n$ tropical vectors in $\mathbb{T}^{(n)}$ are tropically dependent.*

*Proof.* Assume $v_1, \dots, v_{n+1}$ are independent vectors in $\mathbb{T}^{(n)}$ and consider the $(n+1)\times n$ whose rows are these vectors. Extend this matrix by duplicating one of the columns to get a singular matrix, cf. [5, Theorem 2.5], whose rows are tropically dependent by Theorem 3.6, a contradiction. $\square$

Our next goal is to show that the rank of an $m\times n$ matrix is determined as the maximal size of whose maximal nonsingular minor, rather than by a collections of minors of smaller sizes.

**Theorem 3.11.** *An $m\times n$ matrix $A$ of rank $m$, $m \le n$, has an $m\times m$ nonsingular minor $A_{\max}$.*

*Proof.* The case when $m = n$ is obvious by Theorem 3.6, we proceed by induction on $n$. Let $A'$ denote the $m\times(n-1)$ submatrix of $A$ obtained by erasing the last column and let $A''$ be the submatrix of $A$ obtained by erasing the first column. Assuming both $A'$ and $A''$ have rank $< m$, we aim for a contradiction.

Throughout this prove, to make the exposition clearer, we often use matrix products to describe sums; for example we write $(a_1, \dots, a_n)(b_1, \dots, b_n)^{\mathrm{t}}$ for the sum $\bigoplus_i a_i b_i$.

Denoting the rows of $A'$ as $r'_i$, since $\mathrm{rk}(A') < m$, there are $\alpha'_1, \dots, \alpha'_m \in \bar{\mathbb{R}}$, not all of them $-\infty$, such that

$$\alpha'_1 r'_1 \oplus \cdots \oplus \alpha'_m r'_m \in \bar{\mathbb{U}}^{(n-1)}.$$

We write $\bar{\alpha}'$ for the $m$-tuple $(\alpha'_1, \dots, \alpha'_m)$ and define $\bar{\alpha}''$ by the same way for $A''$.

We show that there are $\mu', \mu'' \in \bar{\mathbb{R}}$, for which $\bar{\beta} = \mu'\bar{\alpha}' \oplus \mu''\bar{\alpha}''$ determines a dependence on $A$. We also need to verify that each entry of $\bar{\beta}$ is in $\bar{\mathbb{R}}$.

Let $r_i$ denote the $i$'th row of $A$, $c_j$ denote the $j$'th column of $A$, and write

$$(\mu', \mu'')\begin{pmatrix} - & \bar{\alpha}' & - \\ - & \bar{\alpha}'' & - \end{pmatrix}\left(\begin{array}{c|ccc|c} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,j} & \cdots & a_{m,n} \end{array}\right) = (b_1, \dots, b_n). \tag{8}$$

Since $c_j$, for $j = 2, \dots, n-1$, is a column of both $A'$ and $A''$, and thus $(\bar{\alpha}')(c_j) \in \bar{\mathbb{U}}$ and $(\bar{\alpha}'')(c_j) \in \bar{\mathbb{U}}$, it is clear that $b_j$ is ghost for each $j = 2, \dots, n-1$. So, by leaving only the first and the last column of $A$,

we reduce Formula (8) and write

$$(9) \qquad (\mu',\mu'')\begin{pmatrix} - & \bar\alpha' & - \\ - & \bar\alpha'' & - \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,n} \\ \vdots & \vdots \\ a_{m,1} & a_{m,n}\end{pmatrix} = (\mu',\mu') \overbrace{\begin{pmatrix} (\bar\alpha')(c_1) & (\bar\alpha')(c_n) \\ (\bar\alpha'')(c_1) & (\bar\alpha'')(c_n)\end{pmatrix}}^{=B}.$$

It easy to see that $B$ is singular, just expand $|B|$ to get

$$|B| = (\bar\alpha')(c_1)(\bar\alpha'')(c_2) \oplus (\bar\alpha'')(c_1)(\bar\alpha')(c_n) = \bigoplus_{i,j} \alpha'_i a_{i,1}\alpha''_j a_{j,n} \oplus \bigoplus_{i,j} \alpha''_i a_{i,1}\alpha'_j a_{j,n},$$

and thus $|B| \in \bar{\mathbb{U}}$. Therefore, by Theorem 3.6, $B$ has rank $< 2$ and whose rows are tropically dependent; this means that there are $\mu'$, $\mu''$ in $\mathbb{R}$ for which $(\mu',\mu'')B \in \bar{\mathbb{U}}^{(2)}$.

Assume there is $i$ for which $\beta_k a_{1,i} \succ \bigoplus_{h\neq i}\beta_h a_{1,h}$, where $\beta_i \in \bar{\mathbb{U}}$. But, $\beta_i = \mu'\alpha'_i + \mu''\alpha''_i$ and, by hypothesis on $A'$, there is $\alpha'_k$, $k \neq i$, for which $\alpha'_k a_{1,k} = \alpha'_i a_{1,i}$, equivalently $\mu'\alpha'_k a_{1,k} = \mu'\alpha'_i a_{1,i}$. Therefore, $\beta_i$ can be replaced by $\pi(\beta_i)$. By interchanging $\alpha'_k$ by $\alpha''_k$, and taking all indices with respect to $A''$, the same argument is applied to the column $c_n$.

This shows that $\pi(\bar\beta)$ determines a tropical dependence on the rows of $A$, a contradiction to the data $\mathrm{rk}(A) = m$. Thus, either $A'$ or $A''$ has rank $m$, and by the induction hypothesis has an $m \times m$ nonsingular minor, also a minor of $A$. □

**Corollary 3.12.** *An $m \times n$ matrix $A$ has rank $k$ iff its maximal nonsingular minor is of size $k \times k$.*

*Proof.* $A$ can not have a minor $A_K$ of rank grater than $k$, since otherwise $\mathrm{rk}(A)$ would be grater than $k$. The proof is then completed by Theorem 3.11 applied to $A_K$. □

**Corollary 3.13.** *The rank of a matrix and the rank of its transpose are the same.*

*Proof.* The rank of $A$ and $A^{\mathrm{t}}$ are both equal to the size of the maximal nonsingular minor. □

**Corollary 3.14.** *The rank of a matrix is equal to size of a maximal independent subset of its columns.*

3.2. **Relations to former settings.** Recall that for real matrices our definition of singularity coincides with the known definition for matrices over $(\bar{\mathbb{R}}, \max, +)$, cf Remark 2.3. In [2], Develin, Santos, and Sturmfels, define the tropical rank of an $n \times n$ matrix $A$ over $(\bar{\mathbb{R}}, \max, +)$ to be the largest integer $k$ such that $A$ has a $k \times k$ nonsingular minor, we denote this type of rank by $\mathrm{rk}_D(A)$ and the corresponding nonsingular minor of maximal size by $A_{\max}$. To emphasize, $\mathrm{rk}_D(A)$ is given only for matrices with real entries without any notion of linear dependence. Our work bring in the notion of linear dependence, and in the light of Corollary 3.12 we have:

**Proposition 3.15.** *When $A$ is a real matrix, i.e. $A \in M_{n\times n}(\bar{\mathbb{R}})$, the tropical rank as in Definition 3.1 coincides with that of Develin, Santos and Sturmfels, i.e.*

$$\mathrm{rk}_D(A) = \mathrm{size}(A_{\max}) = \mathrm{rk}(A),$$

*where $A_{\max}$ is a nonsingular minor of maximal size of $A$.*

*Proof.* Immediate by Corollary 3.7 and Corollary 3.12. □

Therefore, concerning real matrices, our rank preserves also the known relation to Barvinok and Kapranov ranks [2], denoted respectively as $\mathrm{rk}_B(A)$ and $\mathrm{rk}_K(A)$, that is

$$\mathrm{rk}(A) \ \le\ \mathrm{rk}_K(A) \ \le\ \mathrm{rk}_B(A),$$

for each $A \in M_{n\times n}(\bar{\mathbb{R}})$.

**Remark 3.16.** *The computation of each of the above ranks for matrices over $(\bar{\mathbb{R}}, \max, +)$ has been proven to be $NP$-complete* [7]. *Thus, in the view of Proposition 3.15, computating our rank is $NP$-complete as well.*

The below definition and proposition were introduced and proven in [1], are applied only to square matrices defined over "pure reals", i.e. non of the matrix entries is $-\infty$.

**Definition 3.17.** *The columns of an $n \times n$ (pure real) matrix $A$ are* ***strongly linearly independent*** *if there is a column vector $v \in \mathbb{R}^{(n)}$ such that the tropical linear system $Ax = v$ has a unique solution $x \in \mathbb{R}^{(n)}$. A square matrix is* ***strongly regular*** *if its columns are strongly linearly independent.*

**Proposition 3.18.** *For a square (pure real) matrix, strongly regular and tropically nonsingular are equivalent.*

Note that according to Definition 3.17, strongly independence is based on both the existence of a solution and on its uniqueness. Using this notion of dependence, the development becomes very difficult and not intuitive.

**Corollary 3.19.** *The tropical rank of a pure real matrix equals the largest size of a strongly linearly independent subset of its columns.*

*Proof.* Immediate, by Proposition 3.18 and the equality of $\mathrm{rk}_D(A) = \mathrm{rk}(A)$ for real matrices. □

## 4. Solutions of homogeneous linear systems

Recall that the zero set of a polynomial $f \in \mathbb{T}[\lambda_1, \dots \lambda_n]$ in $n$ indeterminates $\lambda_1, \dots, \lambda_n$ is defined as

$$Z(f) = \{\mathbf{a} \in \mathbb{T}^{(n)} \mid f(\mathbf{a}) \in \bar{\mathbb{U}}\},$$

where $\mathbf{a}$ stands for $(a_1, \dots, a_n)$, cf. [4, Defintion 2.2]. A polynomial $f$ is said to be homogenous if all of whose monomials are of the same degree.

Using this notion of zeros, we say that a system $S$ of $m$ homogenous linear equations

$$\begin{array}{ccccccc} f_1 & = & a_{1,1}\lambda_1 & \oplus & \cdots & \oplus & a_{1,n}\lambda_n, \\ \vdots & & \vdots & & & & \vdots \\ f_m & = & a_{m,1}\lambda_1 & \oplus & \cdots & \oplus & a_{m,n}\lambda_n, \end{array} \tag{10}$$

has a solution if all equations have a common zero, i.e. there exist $\mathbf{a} \in \mathbb{T}^{(n)}$ such that $f_i(\mathbf{a}) \in \bar{\mathbb{U}}$, for all $i = 1, \dots, m$. When $\mathbf{a} \in \mathbb{R}^{(n)}$ we say that the solution is pure real.

**Remark 4.1.** *Any $\mathbf{a} \in \bar{\mathbb{U}}^{(n)}$ is also a solution of a system* (10)*. We can also also have mixed solutions, these are solutions for which $\mathbf{a}$ has entries in both $\mathbb{R}$ and $\bar{\mathbb{U}}$.*

As usual a system $S$ of the form (10) can be written in matrix terms as $A_S\Lambda^{\mathrm{t}}$, where $A_S$ is the $m \times n$ coefficients matrix of the system $S$ and $\Lambda = (\lambda_1, \dots, \lambda_n)$.

**Theorem 4.2.** *A system $S$ of $n$ homogenous linear equations has a pure real solution iff the corresponding coefficient matrix $A_S$ is singular.*

*Proof.* Obvious by Corollary 3.14. □

Note the a system $S$ with $A_S$ nonsingular can also have mixed solutions.

## References

[1] P. Butkovic and F. Hevery. A condition for the strong regularity of matrices in the minimax algebra. *Discrete Appl. Math.*, pages 209–222, 1985.
[2] M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. *Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge*, 53:213–242, 2005. (Preprint at arXiv:math.CO/0312114).
[3] A. M. Gibbons. *Algorithmic Graph Theory.* Cambridge Univ. Press, Cambridge, UK, 1985.
[4] Z. Izhakian. Tropical algebraic sets, ideals and an algebraic nullstellensatz. *International Journal of Algebra and Computation*, to appear. (Preprint at arXiv:math.AC/0511059,2005).
[5] Z. Izhakian. Tropical arithmetic and algebra of tropical matrices. *Communincation in Algebra*, to appear. (Preprint at arXiv:math.AG/0505458,2005).
[6] Z. Izhakian and L. Rowen. Supertropical algebra. Preprint, 2007.
[7] K. H. Kim and F. W. Roush. Kapranov rank vs. tropical rank. *Proc. Amer. Math. Soc*, 134:2487–2494, 2006. Preprint at arXiv:math.CO/0503044.

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